RESEARCH STATEMENT

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1. Introduction

The discovery of the Jones polynomial in 1984 ignited a flurry of activity in the study of knot polynomials. This renewed interest was compounded in 1999 when Khovanov homology was introduced – the first example of a *categorification* of a knot invariant. Khovanov homology associates to each link a bigraded abelian group whose graded Euler characteristic is the (unnormalized) Jones polynomial of the link. Khovanov homology is known to detect the unknot and a handful of other knots and links, and provides various concordance invariants, including the potent s-invariant.

Heegaard Floer homology, a package of invariants for closed, oriented 3-manifolds, presented another major step in this quest for categorification. The generalization of Heegaard Floer homology to knots, knot Floer homology, also takes the form of a bigraded abelian group (in its simplest form), and categorifies the Alexander polynomial. Knot Floer homology is a powerful invariant: it detects the genus and fiberedness of a knot, provides bounds on the slice genus of knots, and provides concordance invariants, among other results.

My research lies in low-dimensional topology, mainly focused on the above two link homology theories and the connections between them. Much of my work is motivated by a desire to combine computation with theory to make progress on open questions about knots and 3- and 4-manifolds. This has naturally led me to work with tangle Floer homology, which breaks up computations of knot Floer homology to pieces that are feasible for computers, providing me with computational evidence and tools to investigate these questions. I find that even in situations where the computation does not advance a proof, it aids in intuition and fact finding, so is still valuable.

2. Past Results and Proposed Research

2.1. Knot Floer homology.
The first of my projects is an extension of grid homology for lens space links to integer coefficients. After some background on the general theory, I will detail some of my results, as well as future directions and applications of this research.

Motivated by gauge theory, Heegaard Floer homology is a package of invariants introduced by Ozsváth and Szabó in [19]. To a closed, oriented 3-manifold M, Heegaard Floer homology associates a chain complex, whose definition depends on a choice of Heegaard diagram for M as well as a number of analytic choices. The homotopy type of this chain complex is an invariant of the manifold [19].

Independently, Ozsváth and Szabó [17] and Rasmussen [24] generalized this to an invariant of links in 3-manifolds. The introduction of a link L gives a filtration on the Heegaard Floer homology of the 3-manifold, and the filtered homotopy type of this complex is an invariant of the link, as is the homology of the associated graded complex. The knot Floer construction comes in different flavors, the simplest of which is denoted $\widehat{\mathrm{CFK}}$, with homology $\widehat{\mathrm{HFK}}$. A stronger variant (yet harder to compute) is CFK^-, with homology HFK^-, which gives further information about 4-dimensional properties of the knot.

Several years after the introduction of knot Floer homology, Sarkar and Wang showed that the "hat" versions of Heegaard Floer and knot Floer homology admit combinatorial descriptions [26]. This description relies on a Heegaard diagram for the manifold (or the link in the manifold) which is *nice.* They showed that any closed, oriented 3-manifold has a nice diagram, as does any nullhomologous link in such a manifold. Soon after, Manolescu, Ozsváth, Szabó, and Thurston used *grid diagrams* to give a combinatorial description of knot Floer homology for links in S^3 [13]. These grid diagrams provide two advantages over the more general class of nice diagrams: they are uniform, which helps in algorithmic implementation, and they allow for a combinatorial computation of HFK^-, not just $\widehat{\mathrm{HFK}}$. The authors of [13] also provide a standalone combinatorial proof of invariance without appealing to the equivalence with the holomorphic theory. For Heegaard Floer homology, knot Floer homology, and the tangle Floer homology introduced in Section 2.2, there is a "minus" version which is stronger than the "hat" version.

Shortly following, Baker, Grigsby, and Hedden undertook a similar combinatorial program for links in lens spaces [1]. They introduced *twisted toroidal grid diagrams*, the combinatorics of which capture the topology of links in lens spaces $L(p, q)$. Given a twisted toroidal grid diagram $\mathbb{G}$ for a link $L \subset L(p, q)$, they define a chain complex $(C^-(\mathbb{G}), \partial)$. They show that this complex is equivalent to one computing $\mathrm{CFK}^-(L)$, thus proving that the homotopy type of the complex is a link invariant.

In [27], I prove directly the invariance of grid homology for links in lens spaces, providing explicit combinatorially-defined homotopy equivalences, which count various embedded polygons on the twisted toroidal grid diagrams.

Theorem 2.1. *Suppose $\mathbb{G}$ and $\mathbb{G}'$ are two twisted toroidal grid diagrams associated to the same ℓ-component link L in $L(p, q)$. Then $(C^-(\mathbb{G}), \partial_{\mathbb{X}}^-)$ and $(C^-(\mathbb{G}'), \partial_{\mathbb{X}}^{-\prime})$ are quasi-isomorphic as chain complexes over $\mathbb{Z}/2\mathbb{Z}[U_1, \ldots, U_\ell]$ via a combinatorially defined quasi-isomorphism.*

For links in S^3, one application of the combinatorial proof of invariance of [13] is that it allows for a generalization of grid homology to integer coefficients. We find the same is true of links in lens spaces. In [7], Celoria shows that one can associate a chain complex $(C^-(\mathbb{G}), \partial_{\mathbb{X}}^-; \mathbb{Z})$ to a twisted toroidal grid diagram $\mathbb{G}$. In [27], I adapt my proof of invariance over $\mathbb{Z}/2\mathbb{Z}$ and show that the homotopy type of this complex is a link invariant.

Theorem 2.2. *Suppose $\mathbb{G}$ and $\mathbb{G}'$ are two twisted toroidal grid diagrams associated to the same ℓ-component link L in $L(p, q)$. Then $(C^-(\mathbb{G}), \partial_{\mathbb{X}}^-; \mathbb{Z})$ and $(C^-(\mathbb{G}'), \partial_{\mathbb{X}}^{-\prime}; \mathbb{Z})$ are quasi-isomorphic as chain complexes over $\mathbb{Z}[U_1, \ldots, U_\ell]$. Moreover, there is a combinatorially defined quasi-isomorphism between them.*

We note that Theorem 2.2 did not follow from an identification with the holomorphic theory. While there is an extension of the holomorphic theory for links to integral coefficients, it is not known if this extension agrees with the combinatorially defined one. For example, it is not understood how the local orientation systems necessary to define the holomorphic theory with integral coefficients change under combinatorial moves.

There remains an interesting open question of whether there can be torsion found in these integral coefficient grid homology groups. No examples of torsion have been produced yet for links in S^3 or lens spaces, but it is expected that there are links with torsion in their grid homology [18]. There has been some computational study of this, but the complex $C^-(\mathbb{G})$ has a number of generators proportional to $n!$, where n is the size of $\mathbb{G}$, so it is not practical to compute for links with large grid number.

While there are potentially computational techniques that could be used to speed up computation of integer grid homology, there are also mathematical techniques. For knots in S^3, another complex homotopy equivalent to grid homology, but more computationally feasible, was described in [5] and implemented in [9]. I plan to adapt this approach to the case of links in lens spaces, and implement it in Python. The production of a knot with torsion would be interesting, as it would be an example of a knot where the grid homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients does not determine the grid homology with $\mathbb{Z}$ coefficients. Another mathematical approach I plan to take is to adapt invariance proofs to tangle Floer homology and to generalize the notion of sign assignments to this tangle setting, which would allow for a combinatorially defined version of tangle Floer homology with integral coefficients; see Section 2.2.

A second open problem, and the motivation for the introduction of combinatorial lens space grid homology by Baker, Grigsby, and Hedden, is the Berge conjecture.

Conjecture 2.3 ([6])**.** *Suppose that surgery on $K \subset L(p,q)$ yields S^3. Then K has grid number one.*

Any further combinatorial understanding of grid homology for links in S^3 or lens spaces would advance progress on this conjecture. For example, it is known that knot Floer homology detects the unknot, but a proof of this directly from grid homology is an open problem of [18]. The analogue for links in lens spaces is part of the program of [1] towards proving the Berge conjecture.

Conjecture 2.4. *Let $K \subset L(p,q)$ and suppose $\mathrm{rk}(\widehat{\mathrm{HFK}}(K, L(p,q))) = p$. Then K has grid number one.*

2.2. **Tangle Floer homology.** The nice diagrams for manifolds which Sarkar and Wang introduced make the computation of the simplest form of Heegaard Floer homology combinatorial. The cost of this, however, is that they do so by increasing the complexity of the underlying Heegaard diagram, and thus greatly increasing the number of generators of the chain complex associated to the diagram. A natural next step, then, is a bordered Floer theory for bordered manifolds. This theory was constructed by Lipshitz, Ozsváth, and Thurston [11]. To a surface they associate a differential graded algebra (DGA), and to a bordered manifold an $\mathcal{A}_\infty$ module over the DGA associated to boundary. Gluing manifolds along their common boundary corresponds to taking derived tensor product of the corresponding modules, and recovers the Heegaard Floer homology of the closed manifold.

Similarly, a grid diagram representing a link in S^3 comes at the cost of greatly increasing the number of generators of the associated chain complex, compared to an arbitrary Heegaard diagram for the link. Analogous to the theory of bordered Floer homology, Petkova and Vértesi defined tangle Floer homology [21]. To a signed set of points they associate a differential graded algebra, and to a tangle T in $S^2 \times I$ an $\mathcal{A}_\infty$ bimodule over the algebras for the left and right boundaries of the tangle. Concatenating tangles corresponds to taking derived tensor products of these bimodules; for a closed link, the theory recovers knot Floer homology. Like knot Floer homology, tangle Floer homology also comes in a "hat" and "minus" version. The simpler version is denoted $\widehat{\mathrm{CDTA}}(T)$, and it is shown analytically to be an invariant of the tangle, up to $\mathcal{A}_\infty$ homotopy equivalence, in [21].

Question 2.5. *Is there a combinatorial proof that* $\widehat{\mathrm{CDTA}}(T)$ *results in an invariant of the tangle?*

I have made partial progress towards answering the above question, verifying invariance under a subset of tangle moves. For the knot Floer homology of knots and links in S^3 and lens spaces, one advantage of the combinatorial proof of invariance for the "hat" version is that it lends insight into the proof for the "minus" version. We expect the same to be true for tangle Floer homology.

Conjecture 2.6. *The homotopy type of* $\mathrm{CDTA}^-(T)$ *is an invariant of a tangle* T.

Joint with Zachary Winkeler and Shikhin Sethi, I have written Python code[1] to compute these tangle Floer invariants following [21] and [22]. In joint work with Zachary Winkeler, I have recently used that code to provide computational evidence for the following conjecture.

Conjecture 2.7. *Let* $P = \{p_1, \ldots, p_n\}$ *be a set of points in* S^2, *and let* T *be the trivial tangle* $P \times I$ *in* $S^2 \times I$. *Then* $\mathrm{CDTA}^-(T)$ *is quasi-isomorphic to the identity DA bimodule over the differential graded algebra for* P.

A proof of the above would be an important step towards proving the invariance of CDTA^-, since concatenating with a trivial tangle is one of the tangle moves mentioned earlier. On a number of small examples, we have used the code to explicitly compute CDTA^- and construct quasi-isomorphisms with the identity bimodule. We are working on providing a proof of this conjecture.

Another potential application of a combinatorial proof of the invariance of tangle Floer homology would be to define invariants of Legendrian and transverse tangles. On a 3-manifold M, a contact structure is a plane distribution in the tangent space of M satisfying certain non-integrability conditions. The study of contact geometry has physical origins, and as such has applications in many areas of physics. Contact geometry has also been used to great success in low-dimensional topology, including for the proof of Property P by Kronheimer and Mrowka [10], and for the definition of knot invariants by Ng [15, 16]. A knot in a contact 3-manifold is *Legendrian* if the knot is everywhere tangent to the contact structure, and is *transverse* if it is everywhere transverse to the contact structure. An important subfield of contact geometry is dedicated to the study of Legendrian and transverse links in the standard contact S^3, that is S^3 with the plane distribution $\xi_{\mathrm{std}} = \ker(dz + x dy)$. There is an invariant of transverse links denoted $\theta(L)$, with separate constructions given by [20], [12], and [4]. Baldwin, Vela-Vick, and Vértesi proved in [4] that these three constructions all produce the same invariant.

Conjecture 2.8. *To a transverse tangle*[2] T, *there is an element* $\theta(T) \in \widehat{\mathrm{CDTA}}(T)$, *which is an invariant of the transverse tangle. This element behaves nicely with tensor products, recovering* θ *in the case of a knot.*

I have a candidate class for $\theta(T)$, and have proved it is invariant under a subset of transverse tangle moves. A combinatorial proof of the invariance of $\widehat{\mathrm{CDTA}}$ would aid in this, as it provides explicit homtopy equivalences between different diagrams representing the same

[1]github.com/samueltripp/tanglefloer

[2]Here we consider tangles in $S^2 \times I$ or B^3 endowed with a certain contact structure which recovers ξ_{std} under gluing.

tangle via counting embedded polygons, and one would hope it would be easy to track the effect of these maps on $\theta(T)$.

2.3. **Connections between Khovanov homology and knot Floer homology.** The constructions of Khovanov homology and knot Floer homology are quite dissimilar, yet researchers have long noticed strong similarities or parallels between the two. Shortly after the development of knot Floer homology, Rasmussen conjectured the existence of a spectral sequence from Khovanov homology to knot Floer homology [25]. Computational evidence supported this conjecture in the form of a rank inequality, but even this inequality was not proven until 2018, when Dowlin constructed the desired spectral sequence [8]. This connection between the two homologies in the form of a spectral sequence has quickly proven fruitful; for example, it was used to prove that Khovanov homology detects the figure-eight knot [2], the cinquefoil [3], and the $T(2,6)$ torus link [14].

The final project I will discuss here relates to the properties of Dowlin's spectral sequence, and is joint work with Zachary Winkeler. It is known that the E^2 page of Dowlin's spectral sequence is isomorphic to Khovanov homology and the E^∞ page is isomorphic to knot Floer homology, and so both pages are invariants of the knot. One would expect that in fact all pages of Dowlin's spectral sequence are invariants of the knot.

Conjecture 2.9. *Each page E^k for $k \geq 2$ of Dowlin's spectral sequence is an invariant of the knot.*

We have written code[3] computing the pages of this spectral sequence in Macaulay2, but the complex has number of generators exponential in the number of arcs of the knot diagram used in the construction of the spectral sequence, so it is impractical on all but the smallest knots currently. We are working on both mathematical and computational methods for speeding up this code. While we work on computing this spectral sequence, we are also working on proving the above directly. We have proofs that each page is invariant under two of the Reidemeister moves, and are working on resolving this to the full conjecture above.

One application of Conjecture 2.9 is that it would connect invariants which take values in Khovanov homology to those taking values in knot Floer homology. One example is $\psi(K)$, an invariant of a transverse knots, introduced by Plamenevskaya in [23]. This invariant takes the form of a cycle in the Khovanov complex of the transverse knot. Under the isomorphism of [8], $\psi(K)$ is a class on the E^2 page of Dowlin's spectral sequence. If we were able to prove Conjecture 2.9, then the class in knot Floer homology which $\psi(K)$ is sent to under the spectral sequence would be a transverse invariant of the knot. (We would actually have a $\mathbb{Z} \cup \{\infty\}$ family of invariants, one on each page of the spectral sequence.)

Our preferred method would be to define an element $\psi_1(K)$ in the master complex used to construct the spectral sequence which is sent to $\psi_2(K)$ on the E^2 page, representing $\psi(K)$ under the isomorphism of the E^2 page and Khovanov homology. This would give us a more concrete handle on what the class in knot Floer homology would be.

It is possible that this invariant would correspond to the θ invariant from Section 2.2. This equivalence would suggest candidate pairs of knots which ψ might distinguish, and a method of attack for proving such. On the other hand, this invariant might not correspond to the already known transverse invariants in knot Floer homology, and would provide a new invariant of interest. Both prospects are exciting.

[3] github.com/zach-winkeler/khovanov-ss

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